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A receptance-based method for predicting latent roots and critical points in friction-induced vibration problems of asymmetric systems

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Abstract

This paper studies the latent roots and critical points of friction-induced vibration problems in which the stiffness matrix is asymmetric. The asymmetric terms are represented by a parameter or parameters related to the friction coefficient. As the parameter value increases, some latent roots of the asymmetric system change, and even become complex with positive real parts at a critical point, indicating flutter instability. A method is put forward for computing the latent roots and predicting the critical value of this parameter at the flutter instability boundary of the asymmetric system based on the receptance of the symmetric system. When measured receptances of the symmetric system (at those locations where friction forces would be acting in the corresponding asymmetric system) are available, the simulated numerical example shows that this method is efficient.

1. Introduction

The mass matrix and stiffness matrix of engineering structures can be assumed to be symmetric, respectively, positive definite and semi-positive definite in general. The eigensolutions of such structures are extensively studied and the vibration of such systems is stable (or marginally or neutrally stable in control theory). There are, however, engineering problems whose stiffness matrices are asymmetric. Usually the asymmetry is produced not by the structure itself, but by some external loads interacting with the structure [1], such as friction in brake noise problems [2] or airflow in aeroelastic flutter problems [3], though these external factors are considered to be internal to the system. In this paper, a structure is understood to be a building or a machine in a general sense while a system is considered to include a structure and the forces acting on the structure. Symmetric or asymmetric systems mean the stiffness matrices of the systems are symmetric or asymmetric but do not refer to the spatial configuration of the structures.

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In linear or linearised models of friction-induced vibration, friction introduces asymmetric terms into the stiffness matrix [4–10]. Hoffmann et al. [6] demonstrated that as a term Δ proportional to the friction coefficient increased two pairs of purely imaginary latent roots would coalesce (the two frequencies became identical) at $\Delta = 1$ and become complex with one pair having positive real parts when $\Delta > 1$ in their two-degrees-of-freedom system with sliding friction. The critical value of the friction coefficient where flutter-type instability sets in is an important quantity in systems in which friction-induced vibration and noise are major issues and it can be used as a measure of system instability [8]. It is very useful to be able to predict this critical value in brake design. Huang et al. [8] presented a perturbation method for determining the critical friction coefficient for disc brakes based on the eigenvalues and eigenvectors of the elastically coupled system (the symmetric system). The standard approach for brake squeal analysis in automotive industry [2] is computing many complex eigenvalues at gradually varying values of the friction coefficient [11]. Needless to say this is a very expensive process.

This paper presents a method for determining the latent roots of asymmetric systems represented by second-order matrix differential equations and predicting the critical value of the parameter (such as the critical friction coefficient) generating the asymmetry in the stiffness matrix, based on the receptance of the corresponding symmetric systems. The eigenvalue problem and the receptance of the symmetric system are usually well understood and much easier to solve than the asymmetric system. The idea of determining the solutions of a modified system from the receptance of the original (unmodified) system has been used in the context of structural modifications [12–16] and structural control [17]. For a system with a small number of degrees-of-freedom, receptances can be obtained by inverting the dynamic stiffness matrix. For a system with a large number of degrees-of-freedom, however, that is not practical. Then receptances at the right locations of the structure should be measured and used instead. Measurement of receptances for complicated structures such as disc brakes is discussed in Section 5.

2. Theoretical models with asymmetric stiffness matrices

The free vibration problem of a structure may be written as

$$(\mathbf{K} + \lambda^2 \mathbf{M})\boldsymbol{\varphi} = 0 \tag{1}$$

where $\mathbf{M}$ and $\mathbf{K}$ are mass and stiffness matrices, λ and $\boldsymbol{\varphi}$ are called the latent root and latent vector by Lancaster [1]. Usually $\mathbf{M}$ is symmetric and positive definite and $\mathbf{K}$ is symmetric and can be semi-positive definite. Then it is well known that the latent roots are in pairs of purely imaginary numbers in the form of $\pm i\omega_j$ (where ω_j is the j-th natural frequency of the structure and $i = \sqrt{-1}$). Some latent roots are zero only when $\mathbf{K}$ is only semi-positive definite (but not positive definite). It should be said that although λ^2 in Eq. (1) is usually called an eigenvalue, when damping is added λ is also usually called an eigenvalue. To avoid confusion, λ is referred to as a latent root in this paper, as named by Lancaster and advocated by Inman [1].

There are two basic linear models of friction-induced vibration problems. The first one [2,5] may be illustrated by a spring connecting two surfaces in sliding contact shown in Fig. 1. Both surfaces are deformable and must be included in the finite element (FE) model.

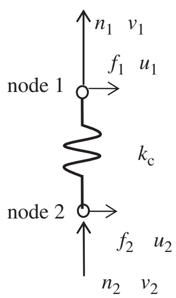

Fig. 1. Contact spring with friction loading.

The element stiffness matrix [2,5] is given in Eq. (2)

$$\begin{Bmatrix} f_1 \\ n_1 \\ f_2 \\ n_2 \end{Bmatrix} = \begin{bmatrix} 0 & \mu k_c & 0 & -\mu k_c \\ 0 & k_c & 0 & -k_c \\ 0 & -\mu k_c & 0 & \mu k_c \\ 0 & -k_c & 0 & k_c \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} \tag{2}$$

where μ is the (kinetic) friction coefficient and k_c is the spring constant representing the contact stiffness between the two interface nodes of the surfaces in contact. f_1, n_1, f_2 and n_2 are the friction forces and normal forces acting on node 1 and node 2, respectively, while u_1, v_1, u_2 and v_2 are the corresponding displacements.

The second basic model is illustrated in Fig. 2 [6], in which a mass is connected by two inclined springs and supported by a vertical spring linking a rigid slider. The mass is allowed to vibrate in the horizontal and vertical directions and the rigid belt moves at a constant speed. It is assumed that sliding always takes place (no sticking) and there is no loss of contact between the slider and the moving belt.

The stiffness matrix is [6]

$$\begin{bmatrix} 2 & 1-\Delta \\ 1 & 2 \end{bmatrix} \tag{3}$$

where $\Delta = \mu k_c$ and the spring constants are $k_1 = 4 - 2\sqrt{3}/3$, $k_2 = 4 + 2\sqrt{3}/3$ and $k_c = 4/3$.

It is clear that both models contain asymmetry in the stiffness matrix. Incidentally, North's model [4] can be interpreted as either a follower force or a friction couple and either introduces asymmetric terms into the stiffness matrix. As μ increases, the asymmetric terms become greater and two pairs of the latent roots in the system shown in Fig. 2 are getting closer and closer and eventually coalesce when the real parts become non-zero (half of them become positive and the other half become negative) [6]. Flutter instability sets in at this point, known as a critical point [18]. This mechanism whereby friction causes flutter instability has been known in the study of brake squeal as mode-coupling [8,9]. Systems with stiffness matrix like Eq. (2) also share the same characteristics. Huseyin [18] made extensive studies of stability of various types of dynamic systems. The focus in this paper, however, is on the type of systems with asymmetric stiffness matrix (or asymmetric coefficient matrix for the zero-order state vector) but no damping. Incidentally, Hochlenert et al. [9] established the equations of motion of a moving beam and a rotating disc with frictional loading, in which the stiffness matrix was also asymmetric and there was a gyroscopic term due to disc rotation. Asymmetric systems with viscous damping or a gyroscopic term are not touched upon in this paper and will be studied later.

It is fairly easy to determine the critical friction coefficient of a simple system with a small number of degrees-of-freedom, like the one shown in Fig. 2, which admits close-form solutions of the latent roots [6]. For systems with a large number of degrees-of-freedom, however, close-form solutions cannot be obtained. If numerical methods of complex eigenvalue analysis are used instead, it would be a very costly process. For each trial value of the friction coefficient, a complex eigenvalue analysis for the whole asymmetric system must be

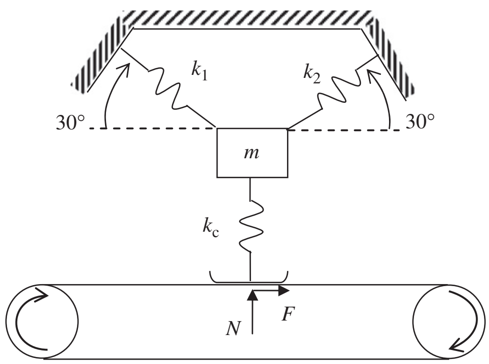

Fig. 2. A two-degrees-of-freedom system with sliding friction (adapted from Ref. [6]).

carried out. A number of such analyses are necessary in order to determine the critical friction coefficient. This paper provides a more efficient method for determining the latent roots and predicting the critical parameter value of the asymmetric system by using the receptances of symmetric system, when the measured values are available.

3. Receptance-based method

Although friction generates asymmetric terms in the stiffness matrix, it should be recognised that the number of such asymmetric terms is very small in comparison with the total number of degrees-of-freedom in a system by the FE method. There are in total about 180,000 degrees-of-freedom among which there are only about 1,600 friction-affected degrees-of-freedom in the FE model of a Mercedes vented disc brake system [7,19], and there are in total 37,100 degrees-of-freedom among which there are only about 900 friction-affected degrees-of-freedom in the FE model of a Mercedes solid disc brake system [20], studied by the first author and his colleagues. Even for the matrix block that corresponds to the friction-affected degrees-of-freedom and hence contains asymmetric terms, there can be a fairly large number of zeros in it.

The equation of motion of an undamped system with frictional stiffness can be written as

$$\mathbf{M}\ddot{\mathbf{x}} + \left(\mathbf{K} + \sum_{i=1}^{n} \mu_i k_{ci} \mathbf{E}_i\right)\mathbf{x} = \mathbf{f} \tag{4}$$

where n is the number of friction-affected degrees-of-freedom in the stiffness matrix, μ_i and k_{ci} are the friction coefficient and contact stiffness of the i-th frictional stiffness term, and matrix $\mathbf{E}_i$ represents the locations of the friction forces, $\mathbf{x}$ and $\mathbf{f}$ are, respectively, the displacement vector and the force vector (other than friction). It must be pointed out that Eq. (4) can represent a variety of applications [1,18], not limited to friction-induced vibration. Therefore the method to be presented is applicable to all applications represented by Eq. (4), even though the example to be used to demonstrate the use of the method is a friction-induced vibration problem.

The Laplace transform of Eq. (4) is

$$\left[\mathbf{M}s^2 + \left(\mathbf{K} + \sum_{i=1}^{n} \mu_i k_{ci} \mathbf{E}_i\right)\right]\mathbf{X}(s) = \mathbf{F}(s) \tag{5}$$

where s is Laplace constant, $\mathbf{X}$ and $\mathbf{F}$ are Laplace transforms of $\mathbf{x}$ and $\mathbf{f}$.

Huseyin [21] showed that the flutter boundary is delimited by those values of μ_i such that

$$\frac{\partial}{\partial s^2}\left[\det\left(\mathbf{M}s^2 + \mathbf{K} + \sum_{i=1}^{n} \mu_i k_{ci} \mathbf{E}_i\right)\right] = 0 \tag{6}$$

While Eq. (6) is useful in theory, it requires the information of the mass, stiffness and friction-generated stiffness matrices of the whole system. In a theoretical analysis, this is usually not a big problem. In practice, however, there are always modelling errors in mass and stiffness matrices. Besides, in vibration testing, it is much easier to measure the receptance than the dynamic stiffness. Moreover, Eq. (6) seems to require the same amount of effort as solving the determinant equation directly from Eq. (5). Huang's perturbation approach [8] is capable of predicting the critical friction coefficient and sensitivity of eigenvalues and eigenvectors. But it involves a sequence of complicated mathematical steps. This paper presents a different and more efficient approach based on the receptance of the symmetric system (that is, the structure without friction, referred to as the statically coupled system in Ref. [8]).

Eq. (5) may be re-written as

$$\left[\left(\mathbf{K} + s^2\mathbf{M} + \sum_{i=1}^{n} \mu_i k_{ci} \mathbf{e}_{p_i} \mathbf{e}_{q_i}^{\mathrm{T}}\right)\right]\mathbf{X}(s) = \mathbf{F}(s) \tag{7}$$

where

$$\mathbf{e}_j = \{0\ 0\ 0\ \ldots\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ \ldots\ 0\ 0\ 0\}^{\mathrm{T}}$$

is a vector whose elements are all zero except its j-th element that is 1. p_i and q_i ($p_i \neq q_i$) are the row and column of the i-th asymmetric term. Multiplying both sides of Eq. (6) by the receptance matrix (or transfer matrix) $\mathbf{H}(s) = (\mathbf{K} + s^2\mathbf{M})^{-1}$ yields

$$\left[\mathbf{I} + \sum_{i=1}^{n} \mu_i k_{ci} \mathbf{H}(s) \mathbf{e}_{p_i} \mathbf{e}_{q_i}^{\mathrm{T}}\right] \mathbf{X}(s) = \mathbf{H}(s)\mathbf{F}(s) \tag{8}$$

where $\mathbf{I}$ is identity matrix having the same dimension as $\mathbf{M}$ and $\mathbf{K}$.

As mentioned before, the number of friction-affected degrees-of-freedom n is much smaller than the total number of degrees-of-freedom. This means that matrix $\sum_{i=1}^{n} \mu_i k_{ci} \mathbf{H}(s) \mathbf{e}_{p_i} \mathbf{e}_{q_i}^{\mathrm{T}}$ has only n columns of non-zero elements. When all the friction-affected degrees-of-freedom are gathered sensibly in forming the stiffness matrix $\mathbf{K}$, only a $n \times n$ matrix fully determines the transfer functions of those relevant elements of $\mathbf{X}$. As it will take many mathematical symbols and much space to show this matrix formulation explicitly, it is decided to demonstrate how the method works through a simulated example in the next section. Coding of the algorithm in MATLAB for general cases is not difficult though.

4. Demonstration and application of the method

A simulated example is shown in Fig. 3, which may be considered an extension to the two-degrees-of-freedom system studied by Hoffmann et al. [6]. The system has four masses with m_1 having a degree-of-freedom in the x direction, m_4 having a degree-of-freedom in the y direction, and m_2 and m_3 having degrees-of-freedom in both directions. The rigid belt moves at a constant speed. The sliding friction at the slider-belt interfaces is governed by Coulomb friction whose static and kinetic friction coefficients are taken to be the same. This is a simplification and prevents stick-slip vibration from happening. The mass and stiffness matrices corresponding to the displacement vector $\mathbf{x} = \{x_1 \ y_4 \ x_2 \ x_3 \ y_2 \ y_3\}^{\mathrm{T}}$ are

$$\mathbf{M} = \begin{bmatrix} m_1 & & & & & \\ & m_4 & & & & \\ & & m_2 & & & \\ & & & m_3 & & \\ & & & & m_2 & \\ & & & & & m_3 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} k_{11} & & k_{13} & & & \\ & k_{22} & & & k_{25} & \\ k_{13} & & k_{33} & k_{34} & k_{35} & \\ & & k_{34} & k_{44} & & k_{46} \\ & k_{25} & k_{35} & & k_{55} & \\ & & & k_{46} & & k_{66} \end{bmatrix}$$

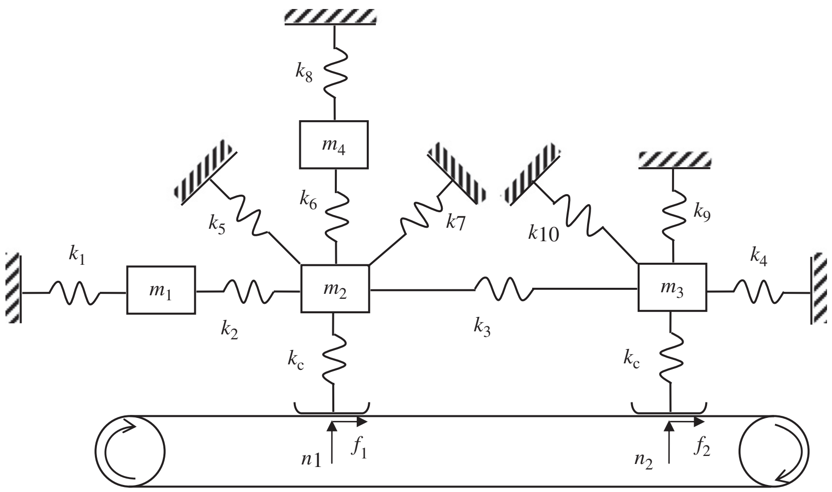

Fig. 3. An asymmetric system of friction-induced vibration.

where

$$k_{11} = k_1 + k_2 \quad k_{13} = -k_2 \quad k_{22} = k_6 + k_8 \quad k_{25} = -k_6 \quad k_{33} = k_2 + k_3 + 0.5(k_5 + k_7)$$
$$k_{34} = -k_3 \quad k_{35} = 0.5(k_7 - k_5) \quad k_{44} = k_3 + k_4 + 0.5k_{10} \quad k_{46} = -0.5k_{10}$$
$$k_{55} = k_c + k_6 + 0.5(k_5 + k_7) \quad k_{66} = k_c + k_9 + 0.5k_{10}$$

In the example, $m_i = m(1, 2, 3, 4)$, $k_i = k(i = 1, 2, 3, 4, 6, 7, 8, 9)$, $k_i = 0.5\,k(i = 5, 10)$. $k_c = 1.1\,k$. $m = 1\,\text{kg}$ and $k = 100\,\text{N/m}$. f_1, n_1, f_2 and n_2 are, respectively, the friction force and (pre-compression) normal force acting at the slider-belt interfaces. The asymmetric part of the stiffness matrix is given by $\mu k_c \mathbf{E}$ where

$$\mathbf{E} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Notice that the asymmetric terms are located at (3, 5) (y_2 affecting x_2 and) and (4, 6) (y_3 affecting x_3), all through friction. Therefore Eq. (8) becomes

$$\left(\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} + \mu k_c \begin{bmatrix} 0 & 0 & 0 & 0 & h_{13} & h_{14} \\ 0 & 0 & 0 & 0 & h_{23} & h_{24} \\ 0 & 0 & 0 & 0 & h_{33} & h_{34} \\ 0 & 0 & 0 & 0 & h_{43} & h_{44} \\ 0 & 0 & 0 & 0 & h_{53} & h_{54} \\ 0 & 0 & 0 & 0 & h_{63} & h_{64} \end{bmatrix} \right) \begin{Bmatrix} X_1 \\ Y_4 \\ X_2 \\ X_3 \\ Y_2 \\ Y_3 \end{Bmatrix} = \mathbf{HF} \tag{9}$$

which can be re-written as

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \mu k_c h_{13} & \mu k_c h_{14} \\ 0 & 1 & 0 & 0 & \mu k_c h_{23} & \mu k_c h_{24} \\ 0 & 0 & 1 & 0 & \mu k_c h_{33} & \mu k_c h_{34} \\ 0 & 0 & 0 & 1 & \mu k_c h_{43} & \mu k_c h_{44} \\ 0 & 0 & 0 & 0 & 1 + \mu k_c h_{53} & \mu k_c h_{54} \\ 0 & 0 & 0 & 0 & \mu k_c h_{63} & 1 + \mu k_c h_{64} \end{bmatrix} \begin{Bmatrix} X_1 \\ Y_4 \\ X_2 \\ X_3 \\ Y_2 \\ Y_3 \end{Bmatrix} = \mathbf{HF} \tag{10}$$

where $h_{ij}(i = 1, 2, \ldots, 6; j = 3, 4)$ are the elements of the receptance matrix $\mathbf{H}$ of the symmetric system. It can be observed that the coefficient matrix of Eq. (10) has least number of non-zero elements in the matrix block of rows 5–6 and columns 5–6, coinciding with the locations of y_2 and y_3 in the displacement vector $\mathbf{x}$. This is actually no coincidence and reflects the nature of the matrix formulation and presents an advantage of the present method.

For the present six-degree-of-freedom example, the analytical expression of the whole $\mathbf{H}$ can be obtained by inverting the dynamics stiffness matrix. In practice, those relevant elements of the frequency response function $\mathbf{H}(\mathrm{i}\omega)$ will be measured and the mathematical expression can be determined through curve-fitting as [22]

$$H_{pq}(\mathrm{i}\omega) = \sum_{j=1}^{n_f} \left(\frac{a_{pq,j}}{\mathrm{i}\omega - s_j} + \frac{\bar{a}_{pq,j}}{\mathrm{i}\omega - \bar{s}_j} \right) \tag{11}$$

so that $H_{pq}(s)$ can be obtained. In Eq. (11), the j-th pair of poles $s_j = (-\xi_j \pm \mathrm{i}\sqrt{1 - \xi_j^2})\omega_j$ where ξ_j is the damping factor (with proportional damping assumed), n_f is the number of frequencies measured (in the range of interest), $a_{pq,j}$ is the associated residue [22], and the overhead bar denotes complex conjugate.

From Eq. (10), one may observe that

$$\begin{bmatrix} 1+\mu k_c h_{35} & \mu k_c h_{45} \\ \mu k_c h_{36} & 1+\mu k_c h_{46} \end{bmatrix} \begin{Bmatrix} Y_2 \\ Y_3 \end{Bmatrix} = \begin{Bmatrix} \sum_{j=1}^{6} h_{5j} F_j \\ \sum_{j=1}^{6} h_{6j} F_j \end{Bmatrix} \tag{12}$$

Note that the receptance matrix **H** is symmetric and in Eq. (12) h_{35}, h_{45}, h_{36} and h_{46} appear in place of h_{53}, h_{54}, h_{63} and h_{64}. The latter should help reveal the relationship between the locations of the friction-affected degrees-of-freedom and those receptance matrix elements involved.

The solution of Eq. (12) is

$$Y_2 = \frac{\sum_{j=1}^{6}[(1+\mu k_c h_{46})h_{5j} - \mu k_c h_{45} h_{6j}]F_j}{(1+\mu k_c h_{35})(1+\mu k_c h_{46}) - \mu^2 k_c^2 h_{36} h_{45}}$$
$$Y_3 = \frac{\sum_{j=1}^{6}[(1+\mu k_c h_{35})h_{6j} - \mu k_c h_{36} h_{5j}]F_j}{(1+\mu k_c h_{35})(1+\mu k_c h_{46}) - \mu^2 k_c^2 h_{36} h_{45}} \tag{13}$$

The latent roots of a system are the poles of its transfer function. Therefore the latent roots are roots of the denominator polynomial of Eq. (13), that is

$$1 + \mu k_c(h_{35} + h_{46}) + \mu^2 k_c^2 (h_{35} h_{46} - h_{36} h_{45}) = 0 \tag{14}$$

which is of course the determinant of the coefficient matrix of Eq. (12), namely, the matrix block formed by rows 5–6 and columns 5–6 of the coefficient matrix in Eq. (10). It should be understood that there is no need to solve Eqs. (10) or (12). In fact, only four (the product of 2 times 2) related receptance matrix elements are involved in Eq. (14), which can be used for determining the latent roots and the critical parameter value. It is also clear from Eq. (10) that there is no need to know the mass and stiffness matrices (nor damping matrix) in order to determine the latent roots and the critical parameter value, if the related receptance elements are known. So a numerical (FE) model (which always contains modelling inaccuracy and uncertain material properties) is not necessary and measured receptances can be used instead [14]. This is a major advantage of all receptance-based methods. Furthermore, measured receptances of only those relevant degrees-of-freedom are required.

For the system shown in Fig. 3, s^2 of the symmetric system and of the asymmetric systems at $\mu = 0.5$ and $\mu = 0.6$ are obtained using the conventional approach to an eigenvalue problem (using MATLAB eig function) and are listed in Table 1 as a comparison. A MATLAB numerical-symbolic programme combined with Maple is coded for the present method. The present method yields results (to four decimal places) identical to those obtained by MATLAB eig function.

It can be observed that s^2 are all negative before instability occurs and a pair of them becomes complex with negative real parts immediately afterwards (however s will have positive real parts in the latter case).

Using MATLAB eig function with a bisection method for searching for the right μ, the critical friction coefficient is found to be $\mu_{cr} = 0.5484$ by the conventional approach.

Table 1
Latent roots and square of latent roots of the original symmetric system and the asymmetric system.

j	1	2	3	4	5	6
s_j^2 ($\mu = 0$)	−97.52	−135.39	−203.07	−280.91	−348.03	−405.07
s_j ($\mu = 0$)	±i9.88	±i11.64	±i14.25	±i16.76	±i18.66	±i20.13
s_j^2 ($\mu = 0.5$)	−107.86	−131.04	−231.32	−255.42	−358.84	−385.53
s_j ($\mu = 0.5$)	±i10.39	±i11.44	±i15.21	±i15.98	±i18.94	±i19.63
s_j^2 ($\mu = 0.6$)	−110.53	−129.52	−243.55+i0.13	−243.55−i0.13	−365.42	−377.44
s_j ($\mu = 0.6$)	±i10.51	±i11.38	±(0.4+i15.61)	±(0.4−i15.61)	±i19.12	±i19.43

By the present method, Eq. (14) becomes

$$\begin{aligned} s^{12} &+ 1470s^{10} + (863850 + 8250\mu)s^8 + (258542500 + 7782500\mu)s^6 \\ &+ (41367375000 + 2612500000\mu + 15125000\mu^2)s^4 \\ &+ (3341250000000 + 366300000000\mu + 6050000000\mu^2)s^2 \\ &+ 106185000000000 + 18370000000000\mu + 605000000000\mu^2 = 0 \end{aligned}$$

This is an even-powered polynomial equation with positive coefficients. It can have either conjugate pairs of roots of purely imaginary numbers or conjugate pairs of roots with positive real parts. The lowest value of μ, which leads to a conjugate pair of roots with a positive real part that crosses zero to positive, is the critical friction coefficient being sought. Using Routh–Hurwitz criterion for the special case that there are no odd-powered terms in the equation [23], three real solutions of μ are found and they are 0.5484, 0.6250 and 0.8064. Therefore $\mu_{cr} = 0.5484$. It is also interesting to observe that these three different critical points correspond to emergence of one pair, two pairs and three pairs of roots with positive real parts (notice that there are in total at most three pairs of latent roots with positive real parts in the current example). Effectively in one go, all critical points are found. This information would be useful if the linear model is a result of linearization and subsequently limit cycles of the corresponding nonlinear system are to be determined.

5. Further discussion of the receptance-based method

As mentioned above, for a complicated system, measured receptances should be used. In the following, the measurement of receptances of one structure is described. Then the receptances of a compound structure formed by connecting two (or more) structures are determined, based on the receptances of the component structures, in the context of a disc brake.

Ewins' monograph [22] gives a comprehensive account of modal testing theory. Several methods for measuring receptances of various structures are available and can be applied to all major brake components. In practice, receptances of the disc are measured using a conventional modal testing technique (usually a hammer impact test, with accelerometers attached to the test structure). So are the receptances of the pads and the rest of the brake. Alternatively, laser vibrometers may be used to obtain receptances, frequencies and modes of the disc or other brake components. If the frequencies and modes of a structure in certain frequency range of interest are available, Eq. (11) shows that the receptances of the structure can be determined.

Although the receptances of a disc brake with the disc being stationary (that is, the symmetric system) should be measured, it is sometimes advantageous to rotate the disc slightly to impart the tangential friction force like prestressing. This will guarantee excitation of the difficult in-plane modes. Here the impulsive friction force serves as an excitation mechanism but not as a cause of instability. Alternatively, non-contact shakers can be used to excite in-plane modes. Chen et al. [24] found that a scanning laser Doppler velocimetry was very effective. It seems that laser metrology is particularly suitable and efficient for measuring frequencies and modes of disc brakes and brake components. A scanning laser vibrometer can capture the whole velocity field of a surface. So the number of measured contact nodes is not a problem. It should be pointed out that measuring velocities and accelerations can also yield receptances (that is, ratios of displacements to forces).

One particular concern might be the contact interface. Contact springs (in the case of node-to-node contact or node-to-surface contact) or contact stiffness (in the case of surface-to-surface contact) is always used at the disc and pads interface in an FE model [2]. A simple way of measuring the contact stiffness is a hammer test for frequencies, as reported in Ref. [24]. More sophisticated techniques include ultrasonic tests [25]. Alternatively, contact area and contact pressure can be measured [20,24] and the values of the contact springs or contact stiffness can be worked out using an inverse method.

The measurement of the receptances of brake components (and even whole brakes) seems to be a lot of extra work. This turns out not to be a problem for brake designers, as car manufacturers and brake suppliers always measure these receptances or frequencies and modes of the brake components and brakes as part of routine. Chen and his colleagues [11,24] at Ford Motor Company and Robert Bosch Corporation reported various experiments of contact area and pressure using sensitive pressure-indicating film, and contact stiffness by a hammer modal test, frequency response functions by hammer and shaker modal tests, and frequencies and modes of brakes and brake components using laser metrology. The receptances of the real disc brakes modelled by the first author's group [7,19] were also measured. One major aim of obtaining the experimental data of the receptances is to validate and update the FE models of real structures [26].

A very useful extension of the receptance-based methods is their application to assembled structures [22] or compound structures [27]. A complex engineering structure may be considered an assembly of simpler component structures, just as a disc brake consists of a disc, two pads, a caliper housing a piston, a carrier cage and other small components. If the receptances at the interface of any two separate structures are known, the receptances of the combined or coupled structure can be determined [27]. For the sake of completeness, the main idea is briefly described below. Readers may consult Ewin [22] for a detailed treatment.

Fig. 4 shows that separate component A and component B are to be connected at nodes a and b by a spring. Suppose the receptances at node a of component A and at node b of component B in the co-linear direction of the spring are, respectively, h_A and h_B and there is an external force f_a acting at node a and an external force f_b at node b. These forces serve to help the subsequent derivation and do not have to be present in reality.

By definition of receptances, one may express the nodal displacements at nodes a and b in terms of the two forces. Further mathematical manipulation of these expressions leads to the point receptances at nodes a and b and the cross receptances from node a to b (or b to a) of the connected structure as

$$h_{AA} = \frac{h_A(1+kh_B)}{1+k(h_A+h_B)} \quad h_{BB} = \frac{h_B(1+kh_A)}{1+k(h_A+h_B)} \quad h_{AB} = h_{\mathrm{BA}} = \frac{kh_Ah_B}{1+k(h_A+h_B)} \tag{15}$$

Ram [27] showed the above idea also worked for a damper or a mass or a combination of them with a spring as a connector. He went on to show that it would work for multiple connectors and two- and three-dimensional connections. In the same vein, those receptances at the disc and pads interface in a disc brake can be determined from modal tests.

In the complex eigenvalue analysis of the dynamics instability of a disc brake, an equilibrium state corresponding to steady-sliding of the disc is first found. Then a perturbation is made about this equilibrium point [5]. The perturbed motion is governed by the linearised model. It is the instability of this perturbed motion that is actually analysed by determining the complex eigenvalues of this motion. Because of this methodology, the contact at the disc and pads interface is assumed to remain unchanged during the perturbation. The nonlinear nature of the contact mechanics is reflected only in the first step of identifying the initial equilibrium point. This is the established methodology used in the brake noise community.

Another matter of interest is presence of damping in real structures. Damping is always a thorny issue. Fortunately, damping in most engineering structures is sufficiently low that it can either be neglected or simplified models can be used [26]. On the whole damping in a disc brake is low enough that proportional damping should be the preferred damping model. When proportional damping is used, the modes of damped structure are the same as those of the undamped structure. As a matter of fact, the method presented in this paper is capable of accommodating non-proportional damping. If one looks at Eq. (14), one can observe that

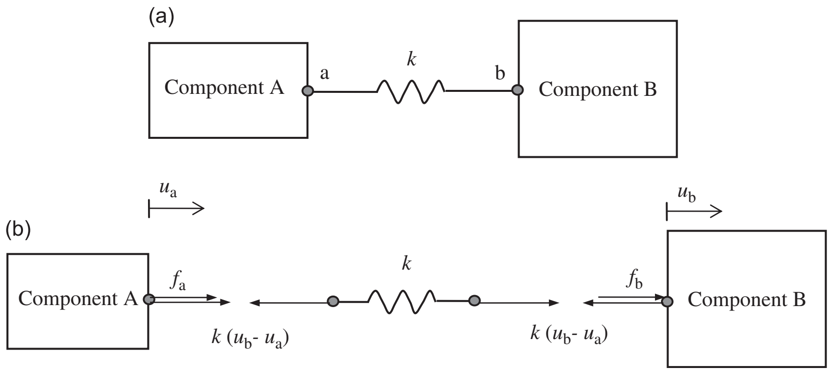

Fig. 4. A structure of two component structures connected by a spring (adapted from Ref. [27]). (a) Components A and B are connected at nodes a and b through a spring stiffness k and (b) free body diagram and forces.

only receptances are needed. There is no assumption made on their mathematical form. Whatever type of damping or whatever magnitude of damping, receptances are measured in the same way and at the same amount of effort. The only restriction is that the system must be linear. Measured receptances always contain damping. So using measured receptances in this method, the predicted latent roots and critical points should be more accurate.

Moreover, although the method presented in this paper is for forward analysis it is capable of assigning latent roots and critical points of damped asymmetric systems by means of structural modifications, when used inversely [28].

Finally, although there are newly developed faster complex eigenvalue extraction methods in recent years, they are not yet adopted by commercial software packages such as ABAQUS, which is the leading software package for brake squeal analysis. The method presented in this paper is efficient when measured receptances are available.

6. Conclusions

This paper presents a method for computing the latent roots and the critical points of asymmetric dynamic systems which have an asymmetric stiffness matrix but no damping. The method requires the knowledge of the receptance elements of only those friction-affected degrees-of-freedom of the symmetric system, which has the advantage that measured receptances can be used directly and a theoretical model, though useful, is not required. All critical points corresponding to the emergence of different numbers of pairs of unstable latent roots can be found in one solution process of a single polynomial equation in the parameter of interest (the friction coefficient in the example).

As the efficiency of the method relies on the knowledge of measured receptances, their measurement in simple structures and compound structures is also discussed, in particular in the case of disc brakes. It is also explained that when damping is present the method still works.

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